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Algebra 1

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Quiz #2

Problems:

1. (30 pt) Let $F : \mathbb{Z} \to \mathbb{Z}$ be a function defined as $F(x) = 10x$.
   (a) Prove that F is a group homomorphism.
   (b) Find $Ker(F)$, what can you say about F?
   (c) Find $Im(F)$.

   Solutions:

   (a) Easy: prove that $F(x+y) = F(x) + F(y)$ for any $x, y \in \mathbb{Z}$.
   (b) $Ker(F) = \{x \in \mathbb{Z} | F(x) = 0\} = \{x \in \mathbb{Z} | 10x = 0\} = \{0\}$. So F is injective.
   (c) $Im(F) = \{F(x) | x \in \mathbb{Z}\} = \{10x | x \in \mathbb{Z}\} = 10\mathbb{Z}$.

2. (10 pt) Let $\phi : G \times X \to X$ be an action of group G on the set $X = G$ given by conjugation, that $\phi(g, x) = gxg^{-1}$. Suppose x is a fixed point of this action, that means that the conjugacy class of x is equal to $\{x\}$ (a single point). Prove $x \in Z(G)$, i.e. x is in the center. **Solutions:**

$$Cx = \{x\} \Leftrightarrow gxg^{-1} = x, \ for\ any\ g \in G \Leftrightarrow gx = xg, \ for\ any\ g \in G \Leftrightarrow x \in Z(G)$$

3. (10 pt) Use the First Isomorphism Theorem to show that $(\mathbb{Z}/30\mathbb{Z})/ < [5] > \simeq \mathbb{Z}/5\mathbb{Z}$ (Note: $(\mathbb{Z}/30\mathbb{Z})/ < [5] >$ means the group $(\mathbb{Z}/30\mathbb{Z})$ quotiented by its subgroup generated by $[5]$ the class of 5).
   Solutions:
   Consider the map $\phi : \mathbb{Z}/30\mathbb{Z} \to \mathbb{Z}/5\mathbb{Z}$ sending $[a]_{30}$ to $[a]_5$. We can show it is a well define morphism, that is surjective, clearly for any $[a]_5 \in \mathbb{Z}/5\mathbb{Z}$, we can take $[a]_{30}$ has the preimage, indeed by definition, $\phi([a]_{30}) = [a]_5$. Note that $[a]_5 = [0]_5$ if and only if $5|a$. Hence, the kernel of ϕ consist of all $[a]_{20}$ for which $5|a$, that is,

$$Ker(\phi) = \{0, 5, 10, 15, 20, 25\} =< [5] > \subseteq \mathbb{Z}/30\mathbb{Z}$$

   It follows by the First Isomorphism Theorem that

$$(\mathbb{Z}/30\mathbb{Z})/ < [5] > \simeq (\mathbb{Z}/5\mathbb{Z})$$